Linear Regression

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Returning to the Road Map

Most policy research involves deceptively simple steps:

1. Define the question you would like answered.
2. State hypotheses about the answer to the question.
3. Collect data that can answer the question (convenience samples, random samples, stratified or multistage samples).
4. Calculate measures to test hypotheses put forward about the relationship of interest (measures of central tendency, measures of spread, test statistics).
5. Organize and report results (graphs, tables, interpretations of measures).

Focusing on Steps 2 and 4: Hypothesis Testing

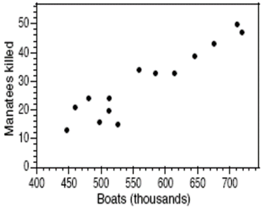
1. State the null and alternative hypotheses and α level of significance
   - Null is a *status quo* assumption about the world you are testing with your sample of data. Stated as $H_o : \mu = X$ where X is an assumption about the true value of μ
   - Alternative is *your* assumption about the world you are testing with your sample of data. Generally, the alternative hypothesis takes the form of $H_1 : \mu \neq X$, $H_1 : \mu > X$, or $H_1 : \mu < X$.
   - α is a probability, from 0 to 1, that represents the maximum threshold of a p-value you will accept for rejecting the null. Conventionally, social scientists use $\alpha = 0.05$.
2. Calculate t-statistic to test the null hypothesis
   - $t = \frac{(\overline{X} - \mu)}{\frac{s}{\sqrt{n}}}$ using the mean, standard deviation, and n from your sample and plugging in your null hypothesis for μ
3. Use the absolute value of t to find the p-value.
4. Compare the p-value to α; if $p < \alpha$, reject the null hypothesis.

Reminders

- Hypothesis testing is always about whether a *statistic* (e.g, $\overline{X}, \overline{X}_1 - \overline{X}_2$) accurately reflects a *parameter of interest* (e.g., $\mu, \mu_1 - \mu_2$).
- A *parameter* can be the value of a single variable in a typical observation in a population OR the typical relationship between two variables in a typical observation in a population.
- The logic of hypothesis testing for a relationship between two variables is very similar to the logic of testing a statistic from a sample - how confident are we that our estimate of the relationship is not due to random chance?

Linear Regression Setup

- Linear regression continues our effort at the same goal we've had in previous weeks: using a sample to estimate a population parameter (thus far, μ) and test hypotheses about the population parameter.



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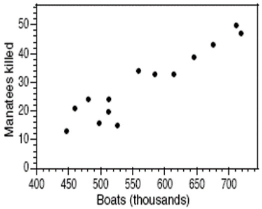

- Now we move to a parameter that captures a relationship between two variables in a population, similar to two-sample hypothesis testing. We've seen scatterplots of x and y before. They also come from random samples and change across samples.

Linear Regression Setup

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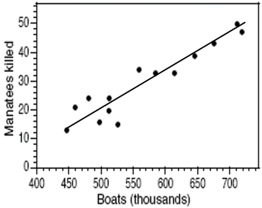

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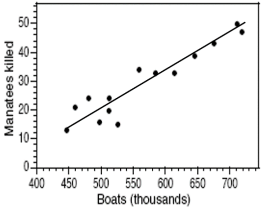

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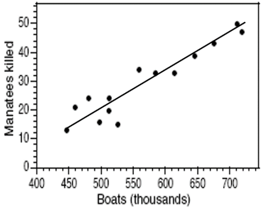

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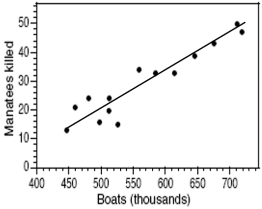

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- Mathematically expressed: $\mu_y = \beta_0 + \beta_1 x$
- We also want to know if β_x, the relationship observed, is statistically significant (i.e., not attributable to chance or sampling error).

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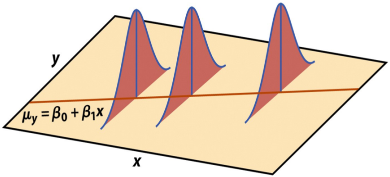

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  - b_0 unbiased estimate for intercept β_0
  - b_1 unbiased estimate for slope β_1

Estimating parameters

Calculating the best fit line ourselves would involve first calculating the slope:

$$\beta_1 = \frac{\sum(x_i - \overline{X})(y_i - \overline{Y})}{\sum(x_i - \overline{X})^2} \tag{1}$$

...and then using the basic form of a line to calculate the intercept:

$$\beta_0 = \overline{Y} - \beta_1\overline{X} \tag{2}$$

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- s provides an unbiased estimate of the regression standard deviation σ, which we can use for inference about the mean population response μ_y.

Regression Standard Errors, continued

The formula is similar for the standard error of the slope (β_1), only the regression standard error (s_{reg}) is divided by the square root of the squared residuals of X:

$$SE_{b1} = \frac{s_{reg}}{\sqrt{\sum(x_i - \overline{x})^2}} \tag{4}$$

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- Note that t* is the t-critical value for the t(n-2) distribution with area C between -t* and $+$t*.

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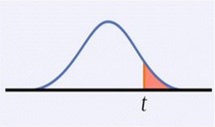
- Finally, as before, we compare the p-value to our α threshold and infer whether β_1 is significantly different from 0 given our sample.

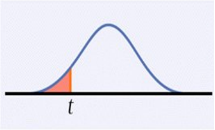
Significance test for the slope

Visually:

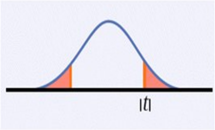
$H_a: \beta_1 > 0$ is $P(T \geq t)$



$H_a: \beta_1 < 0$ is $P(T \leq t)$



$H_a: \beta_1 \neq 0$ is $2P(T \geq |t|)$



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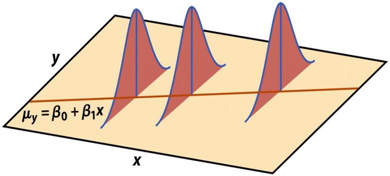

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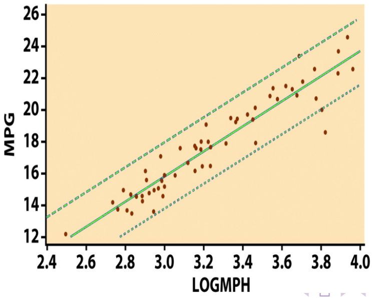

Confidence Intervals for Mean Response (μ_y)

- The confidence interval for μ_y contains, with level C% confidence, the population mean μ_y at a particular level of x.
- The prediction interval contained C% of all the individual values taken by y at a particular value of x.

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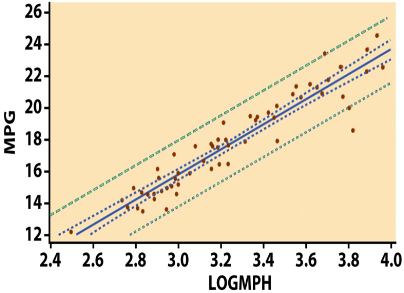

95% prediction interval for $\widehat{y}$ in green
95% confidence interval for μ_y in blue

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- $R^2 = \dfrac{\textit{variation in y caused by x (the regression line)}}{\textit{total variation in observed y values around the mean}}$
- More formally:

$$R^2 = \frac{\sum(\widehat{y}_i - \overline{y})^2}{\sum(y_i - \overline{y}_i)^2} = \frac{SSM}{SST} \tag{7}$$